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Abstract

A graph $G = (V, E)$ is called k-edge rigid (k-edge globally rigid, resp.), if it stays rigid (globally rigid, resp.) after the deletion of at most $k - 1$ edges. We can define k-vertex rigidity and k-vertex global rigidity in a similar manner. It is known that if G is 3-edge rigid (2-edge globally rigid, 2-vertex globally rigid) with $|V| \geq 5$ then $|E| \geq 2|V|$ holds. Furthermore, the graphs that satisfy the edge count with equality are all 4-regular.

In this paper we show that for a 4-regular graph G the properties of 3-edge rigidity, 2-edge global rigidity, and essential 6-edge connectivity are equivalent. By sharpening a result of H. Fleischner, F. Genest, and B. Jackson we give a new inductive construction for the family of 4-regular and essentially 6-edge connected graphs (and hence also for the 4-regular graphs with these rigidity properties). We prove that G is 2-vertex globally rigid if and only if it is 4-vertex connected and essentially 6-edge connected.

We also consider 2-vertex rigid graphs $G = (V, E)$ with minimum size $|E| = 2|V| - 1$ as well as with $|E| = 2|V|$. In the former case we use our results on essentially 6-edge connected graphs to develop a new inductive construction, complementing an earlier, different construction of B. Servatius. In the latter case we characterize the edge pairs of G whose deletion preserves rigidity, and use this result to verify the correctness of a construction of 3-vertex rigid graphs on $|V| \geq 6$ vertices and with $|E| = 2|V| + 2$ edges, proposed by S.A. Motevallian, C. Yu, and B.D.O. Anderson.

1 Introduction

A d-dimensional *framework* (or geometric graph) is a pair (G, p), where G is a simple graph and $p : V(G) \to \mathbb{R}^d$ is a map. We also call (G, p) a *realization* of G in $\mathbb{R}^d$. The

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length of an edge uv in the framework is defined to be the distance between the points $p(u)$ and $p(v)$. The framework is said to be *rigid* in $\mathbb{R}^d$ if every continuous motion of its vertices in $\mathbb{R}^d$ that preserves all edge lengths preserves all pairwise distances. It is *globally rigid* in $\mathbb{R}^d$ if the edge lengths uniquely determine all pairwise distances. A relization (G,p) is *generic* if the set of the $d|V(G)|$ coordinates of the vertices is algebraically independent over the rationals. It is known that for generic frameworks rigidity and global rigidity in $\mathbb{R}^d$ depends only on the graph of the framework, for every $d \geq 1$. So we may call a graph G *rigid* (resp. *globally rigid*) in $\mathbb{R}^d$ if every (or equivalently, if some) d-dimensional realization of G is rigid (resp. globally rigid). For $d = 1, 2$ good characterizations for the rigid and globally rigid graphs in $\mathbb{R}^d$ are available (see the next section). Finding similar characterizations for $d \geq 3$ is a major open problem in rigidity theory. We shall only consider the case $d = 2$ and omit the reference to the dimension in the rest of the paper. The reader is referred to [9, 14] for more details on the theory of rigid and globally rigid frameworks and graphs.

Rigid and globally rigid graphs occur in several applications, including sensor network localization, molecular conformation, formation control, and statics. In some applications it is desirable to identify or construct graphs that remain rigid or globally rigid after the removal of some vertices or edges. This motivates the next definitions.

We say that a graph $G = (V,E)$ is *k-vertex rigid* (resp. *k-vertex globally rigid*) if $G - X$ is rigid (resp. globally rigid) for all $X \subseteq V$ with $|X| \leq k-1$. A graph $G = (V,E)$ is said to be *strongly minimally k-vertex rigid* (resp. *strongly minimally k-vertex globally rigid*) if it is k-vertex rigid (resp. k-vertex globally rigid) and no graph on $|V|$ vertices with less than $|E|$ edges satisfies this property. We can define (strongly minimal) k-edge rigidity and k-edge global rigidity in a similar way, by the deletion of edge sets, rather than vertex sets. The basic problem arising in this setting is to find the best possible lower bound, in terms of k and $|V|$, for the size of the strongly minimal graphs. It amounts to proving a lower bound and finding an infinite family of such graphs that attains this bound. In two dimensions this problem has been solved for all $k \geq 1$ and for each of the four versions (rigid or globally rigid, vertex or edge deletion), see [8]. A related problem is to obtain a simpler characterization and-or an inductive construction that generates every strongly minimal graph from a small base graph. We shall focus on this problem in five special cases, with $k \in \{2, 3\}$.

It is well known that the number of edges in a strongly minimally rigid (also called minimally rigid) graph on $|V|$ vertices is equal to $2|V|-3$. In the case of 2-edge rigidity and global rigidity the extremal number is $2|V| - 2$, assuming $|V| \geq 4$. Inductive constructions are also available: the so-called Henneberg construction of minimally rigid graphs is a basic result in rigidity theory. Constructions for strongly minimally 2-edge rigid and globally rigid graphs (i.e. for rigidity circuits and 3-connected rigidity circuits, resp.) can be found in [1].

It has been verified that the size of the strongly minimally 2-vertex globally rigid [16], 2-edge globally rigid [8], and 3-edge rigid [8] graphs on $|V| \geq 5$ vertices is equal to $2|V|$, and every strongly minimal graph is 4-regular. Our goal is to obtain simpler characterizations, in terms of connectivity properties, and inductive constructions for these families. We also show that our results give rise to a new inductive construction for strongly minimally 2-vertex rigid graphs. These graphs satisfy $|E| = 2|V| - 1$ and are almost 4-regular. A different inductive construction was given by B. Servatius in [15].

In the last section we consider 2-vertex rigid graphs with $|E| = 2|V|$ and characterize the edge pairs whose deletion preserves rigidity. By using this result we can show that the strongly minimally 3-vertex rigid graphs on $|V| \geq 6$ vertices satisfy $|E| = 2|V| + 2$. We rely on a lower bound and construction from a paper by S.A. Motevallian, C. Yu, and B.D.O. Anderson [12]. Here we complete the proof of correctness of their construction. We also provide a different, simpler construction.

Notation

Let $G = (V, E)$ be a graph and $X \subseteq V$. The set of *neighbours* of X, that is, the set of vertices in $V \setminus X$ which are connected to X by at least one edge, is denoted by $N_G(X)$. The set of edges of G with exactly one end-vertex in X is denoted by $\delta_G(X)$. Edge sets of this form are called *edge cuts*. Clearly, $\delta_G(X) = \delta_G(V \setminus X)$. If $\min\{|X|, |V \setminus X|\} \geq 2$ then we say that $\delta_G(X)$ is a *non-trivial* edge cut. We define $d_G(X) = |\delta_G(X)|$. For a singleton $X = \{v\}$ we simply write $d_G(v)$, which is the *degree* of v in G. We shall use the notation $S_G(X)$ for the set, and $s_G(X)$ for the number of edges incident with X. The number of edges of G with both end-vertices in X is denoted by $i_G(X)$. Thus $s_G(X) = i_G(X) + d_G(X)$. The subgraph of G induced by X is denoted by $G[X]$.

2 Rigid and globally rigid graphs

Let $G = (V, E)$ be a graph. It is well-known that if G is rigid then $|E| \geq 2|V| - 3$ holds. Rigid graphs with $|E| = 2|V| - 3$ are called *minimally rigid*. G. Laman gave a combinatorial characterization of minimally rigid graphs by using the following notion. We call G *sparse* if

$$i_G(X) \leq 2|X| - 3 \tag{1}$$

holds for all $X \subseteq V$ with $|X| \geq 2$.

Theorem 1. *[11] Let $G = (V, E)$ be a graph with $|E| = 2|V| - 3$. Then G is rigid if and only if G is sparse.*

It is also known that every rigid graph has a minimally rigid spanning subgraph. This fact and Theorem 1 can be used to characterize rigidity, see e.g. [7]. (We shall not use this more general result in this paper.) Another basic result in rigidity theory provides an inductive construction of minimally rigid graphs by using two operations, called 0-*extension* and 1-*extension*. The first one adds a new vertex v to a graph and two new edges incident with v. The second one deletes an edge xy of G and adds a new vertex v and three new edges incident with v, including vx and vy. See Figure 1. Every minimally rigid graph can be obtained from K_2 by a sequence of such operations, see e.g. [14]. We shall use that if G is rigid and G' is obtained from G by a 0- or 1-extension, then G' is also rigid. A simple corollary of this fact is that if G is 2-edge rigid and G' is obtained from G by adding a new vertex v and at least three new edges incident with v then G' is also 2-edge rigid.

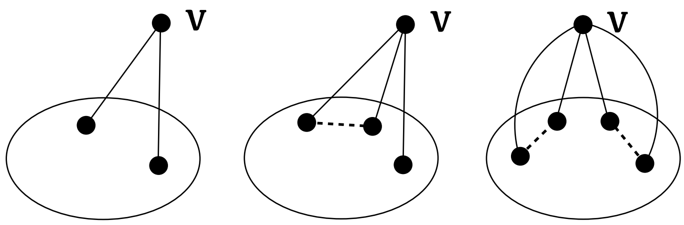

Figure 1: The 0-extension, 1-extension, and 2-extension operations.

The following two lemmas will be used in the last section.

Lemma 2. *Suppose that $G = (V, E)$ is rigid. Then for every $X \subseteq V$ with $|V \setminus X| \geq 2$ we have $s_G(X) \geq 2|X|$.*

Proof. Let $H = (V, F)$ be a minimally rigid spanning subgraph of G. The sparsity of H implies that $i_H(V - X) \leq 2|V - X| - 3$, which gives $s_G(X) \geq s_H(X) \geq |F| - i_H(V - X) \geq 2|V| - 3 - (2|V - X| - 3) = 2|X|$. □

Suppose that $G[X]$ is a tree for some nonempty $X \subseteq V$, for which $d_G(x) = 3$ for all $x \in X$. Then $G[X]$ is called a *cubic subtree* of G.

Lemma 3. *Let $G = (V, E)$ be a graph and let $G[X]$ be a cubic subtree of G for which $G[V \setminus X]$ is rigid. Then for each pair $e, f \in S_G(X)$ we have that (i) $G - e$ is rigid, and (ii) $G - \{e, f\}$ is not rigid.*

Proof. If $|X| = 1$ then (i) follows from the fact that $G-e$ can be obtained from $G[V \setminus X]$ by a 0-extension. Using this as the base case, it is not hard to see by induction on $|X|$ that $G-e$ can be obtained from $G[V \setminus X]$ by a sequence of 0-extensions. This proves (i). Since $G[X]$ is a cubic subtree, we have $s_{G-\{e,f\}}(X) \leq 3|X| - (|X| - 1) - 2 = 2|X| - 1$. By Lemma 2 this implies (ii). □

Global rigidity is characterized by the following result.

Theorem 4. *[5] A graph G is globally rigid if and only if either G is a complete graph on at most three vertices, or G is 2-edge rigid and 3-vertex connected.*

It follows that a 2-edge rigid (or globally rigid) graph on $|V| \geq 4$ vertices has at least $2|V| - 2$ edges. A 2-edge rigid graph $G = (V, E)$ with $|E| = 2|V| - 2$ is called a *rigidity circuit.* It follows from Theorem 1 that a graph G with $|E| = 2|V| - 2$ is a rigidity circuit if and only if it is "minimally non-sparse", that is, every *proper* subset X of V satisfies (1). See Figure 2.

Thus the strongly minimally 2-edge rigid (globally rigid, respectively) graphs on at least four vertices are the rigidity circuits (3-vertex connected rigidity circuits, respectively). Inductive constructions for these two families of graphs were given in [1].

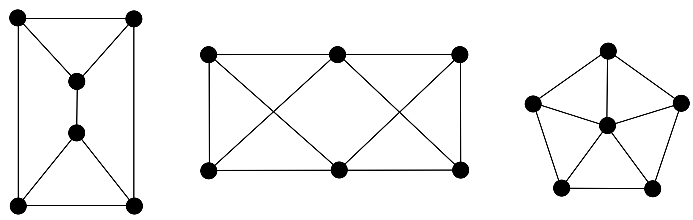

Figure 2: A minimally rigid graph, a rigidity circuit, and a 3-connected rigidity circuit.

3 Essentially 6-edge connected graphs

We say that a graph $G = (V, E)$ is *essentially k-edge connected* if every edge cut of size less than k is trivial. We shall consider essentially 6-edge connected 4-regular graphs. It is easy to see that such a graph on at least four vertices is simple.

Lemma 5. *Let $G = (V, E)$ be an essentially 6-edge connected 4-regular graph with $|V| \geq 5$. Then $G - e$ is 3-vertex connected for all $e \in E$.*

Proof. Suppose that $G - e$ is not 3-vertex connected. Then V can be partitioned into three sets A, S, B such that $|S| \leq 2$, A and B are both non-empty, and there are no edges in $G - e$ from A to B. Since G is 4-regular and simple, it follows that $|A| \geq 2$ and $|B| \geq 2$. Hence $\delta_{G-e}(A)$ and $\delta_{G-e}(B)$ are edge-disjoint non-trivial edge cuts in $G - e$ such that their edges are all incident with S. The essential 6-edge connectivity of G implies that $|\delta_{G-e}(A)| + |\delta_{G-e}(B)| \geq 10$. This contradicts the fact that the total degree of the vertices in S in $G - e$ is at most 8, as $|S| \leq 2$ and G is 4-regular. □

Let G be a graph and let ab, cd be two disjoint edges of G. The *2-extension operation* (on edges ab, cd) adds a new vertex v to G and four new edges va, vb, vc, vd, and deletes the edges ab, cd. See Figure 1. It is easy to see that 2-extension preserves essential 6-edge connectivity as well as 4-regularity (see [10, Lemma 5.1]). The *2-reduction* operation may be viewed as the inverse of 2-extension. It removes a vertex v which has four neighbours a, b, c, d in G and adds two disjoint edges that connect two pairs of its neighbours. In an essentially 6-edge connected graph G we call a vertex v of degree four *2-reducible* if there is a 2-reduction at v for which the resulting graph is essentially 6-edge connected.

Let us call w a *partner* of v (in G) if $vw \in E(G)$ and $N_G(v) - \{w\} = N_G(w) - \{v\}$. It is a symmetric relation. It is easy to see that if G is an essentially 6-edge connected 4-regular graph on at least five vertices and $v \in V(G)$ then (i) v has at most one partner, unless $G = K_5$, (ii) if v has a partner in G then v is not 2-reducible. Fleischner, Genest, and Jackson [2] verified that every vertex which is not 2-reducible must have a partner.

Lemma 6. *[2] Let $G = (V, E)$ be an essentially 6-edge connected 4-regular graph with $|V| \geq 5$ and let $v \in V$. Then v is not 2-reducible if and only if v has a partner.*

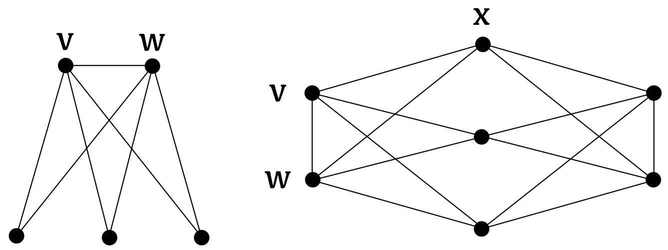

Figure 3: Vertex w is a partner of v (left). A graph in which v and w are not 2-reducible. Vertex x is 2-reducible (right).

As it was pointed out in [10, Theorem 5.2], Lemma 6 can be used to deduce that every essentially 6-edge connected 4-regular graph can be obtained from K_5 by a series of 2-extensions and an additional operation that adds two vertices at a time. We next refine this result, proving that, in fact, every essentially 6-edge connected 4-regular graph can be obtained from K_5 by a series of 2-extensions alone.

Lemma 7. *Let $G = (V, E)$ be an essentially 6-edge connected 4-regular graph with $|V| \geq 6$. Then either $|V| = 6$ and no vertex has a partner, or $|V| \geq 7$ and at least $\frac{3|V|}{7}$ vertices of G have no partners.*

Proof. There is only one 4-regular graph on six vertices (K_6 minus a perfect matching), in which no vertex has a partner. Thus we may assume that $|V| \geq 7$. Suppose that v and w are partners. Let $A = \{v, w\}$, $N_G(A) = \{x, y, z\}$, and $B = V - A - N_G(A)$. We claim that x, y, or z cannot have partners.

By symmetry it suffices to consider $x \in N_G(A)$. Observe that x is connected to B by an edge, for otherwise the 4-regularity of G implies that $\delta_G(A \cup N)$ is a non-trivial edge cut of size at most two. Let $xq \in E$ with $x \in N$ and $q \in B$. Then, since x has neighbours in A as well as in B, the only possible partner of x is one of y or z, say y. But then $\delta_G(A \cup \{x, y\})$ is a non-trivial edge cut of size four, a contradiction. This proves the claim.

Therefore each pair v, w of partners is incident with six edges that connect them to vertices with no partners. So if p vertices have partners then $3p$ edges have this property. The 4-regularity of G implies that there must be at least $\frac{3p}{4}$ vertices with no partners, which implies the lemma. $\square$

The graph on the right in Figure 3 is an essentially 6-edge connected 4-regular graph with exactly $\frac{3|V|}{7}$ 2-reducible vertices.

Theorem 8. *Let $G = (V, E)$ be a 4-regular graph with $|V| \geq 5$. Then G is essentially 6-edge connected if and only if it can be obtained from K_5 by a series of 2-extensions.*

Proof. Sufficiency is easy to check. The proof of necessity is by induction on $|V|$. The statement is trivial if $|V| = 5$, so we may assume that $|V| \geq 6$. By Lemmas 6 and 7 there is a vertex v which is 2-reducible. Let H be an essentially 6-edge connected 4-regular graph obtained from G by a 2-reduction at v. By induction H may be obtained from K_5 by a series of 2-extensions. Extending this series by a single 2-extension that rebuilds G from H, we obtain that G may also be obtained from K_5 by a series of 2-extensions. □

We remark that Theorem 8 can also be deduced from the inductive construction of rigidity circuits given in [1], using [10, Lemma 2.1]. We close this section with a useful lemma.

Lemma 9. *[5] Every rigid graph in $\mathbb{R}^2$ is essentially 3-edge connected.*

4 2-edge global rigidity and 3-edge rigidity

In this section we give a complete characterization of the strongly minimally 2-edge globally rigid and 3-edge rigid graphs.

Theorem 10. *Let $G = (V, E)$ be a 4-regular graph with $|V| \geq 5$. Then the following are equivalent:*
(i) G is essentially 6-edge connected,
(ii) G is 3-edge rigid,
(iii) G is 2-edge globally rigid,
(iv) G can be obtained from K_5 by a sequence of 2-extensions.

Proof. (i) → (ii) Suppose that G is essentially 6-edge connected. It suffices to show that for all edges $e \in E$ the graph $G - e$ is 2-edge rigid. For a contradiction let us assume that $G - e$ is not 2-edge rigid for some $e \in E$. Since $G - e$ has $2|V| - 1$ edges, there exists a subgraph C of $G - e$ which is a rigidity circuit. Let $X = V(C)$ and $Y = V \setminus X$. We must have $Y \neq \emptyset$, for otherwise the 2-edge rigidity of C would imply that $G - e$ is also 2-edge rigid. If $|Y| = 1$, then $Y = \{v\}$ for some $v \in V$ with $d_{G-e}(v) \geq 3$. As the addition of a vertex of degree (at least) three preserves 2-edge rigidity, it follows that $G - e$ contains a 2-edge rigid spanning subgraph, which contradicts our assumption. So we must have $|Y| \geq 2$. In this case the 4-regularity of G and $|E(C)| = 2|V(C)| - 2$ imply that there are at most four edges between X and Y in G. Since $|X| \geq 4$ and $|Y| \geq 2$, it follows that $\delta_G(X)$ is a non-trivial edge cut of size at most four, a contradiction.

(ii) → (iii) Consider a 3-edge rigid graph G and an edge $e \in E$. Clearly, $G - e$ is 2-edge rigid. By Lemma 5 $G - e$ is also 3-vertex connected. Therefore G' is globally rigid by Theorem 4. So G is 2-edge globally rigid, as required.

(iii) → (i) Suppose that G is 2-edge globally rigid. By a result of Hendrickson [4] globally rigid graphs (on at least four vertices) are 2-edge rigid. Thus $G - \{e, f\}$ is rigid for every edge pair $e, f \in E$. We can now deduce from Lemma 9 and the 4-regularity of G is essentially 6-edge connected.

Finally, the equivalence of (i) and (iv) is guaranteed by Theorem 8. □

5 2-vertex global rigidity

In this section we show that 4-regular 2-vertex globally rigid graphs can also be characterized by a connectivity condition. We start with a lemma, which provides a fifth equivalent property of 4-regular graphs (c.f. Theorem 10).

Lemma 11. *Let $G = (V, E)$ be a 4-regular graph with $|V| \geq 5$. Then G is essentially 6-edge connected if and only if $G' = G - v$ is 2-edge rigid for all $v \in V$.*

Proof. Suppose that G is essentially 6-edge connected and let $v \in V$. Consider two edges e, f incident with v in G. Since G is 3-edge rigid by Theorem 10, $H = G - \{e, f\}$ is rigid. We have $d_H(v) = 2$, so G' is also rigid. The edge count $|E(G')| = 2|V(G')| - 2$ now implies that either G' is an M-circuit (in which case we are done, since M-circuits are 2-edge rigid), or it contains a unique M-circuit C as a proper subgraph, induced by a vertex set $X \subset V(G')$. Since G is 4-regular, it follows that C has four vertices of degree three (and all other vertices in C are of degree four) in G'. This implies that $\delta_G(X)$ is a non-trivial edge cut of size at most four, which contradicts our assumption. Next we show sufficiency. For a contradiction suppose that there is a non-trivial edge cut $\delta(X)$ of size (at most) four in G. The 4-regularity of G implies that X (and hence $V - X$) has cardinality at least four. By removing the end-vertex v of some edge $e = uv$ from the edge cut, and another edge $f \in \delta(X)$ which is disjoint from v, we obtain a graph G'' with a non-trivial edge cut of size (at most) two. Thus G'' is not rigid by Lemma 9, a contradiction. □

Necessity in Lemma 11 follows also from [10, Lemma 2.1(ii)]. We can now deduce the main result of this section.

Theorem 12. *Let $G = (V, E)$ be a 4-regular graph. Then G is 2-vertex globally rigid if and only if G is 4-vertex connected and essentially 6-edge connected.*

Proof. First suppose that G is 2-vertex globally rigid and let $v \in V$. Then $G' = G - v$ is globally rigid, and hence G' is 3-vertex connected and 2-edge rigid by Theorem 4. Hence G is 4-vertex connected, and by using Lemma 11, we obtain that it is also essentially 6-edge connected.

Next we prove sufficiency. Suppose that G is 4-vertex connected and essentially 6-edge connected and let $v \in V$. Then $G' = G - v$ is 3-vertex connected and, again by Lemma 11, 2-edge rigid. Thus it is globally rigid by Theorem 4. □

Examples

Let n, k be positive integers with $n \geq k + 2$, $k \geq 2$, and let $C_{n,k}$ be the graph obtained from a cycle on n vertices by adding the two edges that connect a vertex v to its k^{th} neighbors along the cycle (one in each direction), for all v. For example, $C_{n,2}$ is the square of the cycle C_n. See Figure 4. It was pointed out in [16] that the graphs $C_{n,k}$ are 2-vertex globally rigid for $k \in \{2, 3\}$ and for all $n \geq 2k + 1$. We now extend this statement to *every* k.

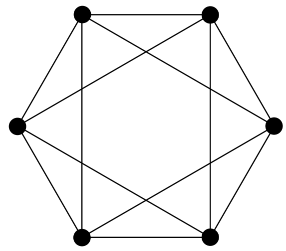

Figure 4: The graph $C_{6,2}$: the square of the six-cycle.

First observe that $C_{n,k}$ is vertex-transitive and, for $n \geq 2k+1$, 4-regular. It is known that a d-regular vertex-transitive graph has vertex-connectivity at least $2(d+1)/3$, see e.g. [3]. Tindell [17] proved that if a vertex-transitive graph G has a non-trivial edge cut of size d, then the vertex set of G can be partitioned into sets of size d such that each member in the partition induces a complete graph K_d. Since $C_{n,k}$ (with $n \geq 2k+1$) has no K_4 subgraphs, we can deduce from the above discussion (by putting $d = 4$) that the graph $C_{n,k}$ is 4-vertex connected and essentially 6-edge connected for all integers n, k with $k \geq 2$ and $n \geq 2k+1$. By using Theorem 12, we may conclude that for $n \geq 2k+1$, the graph $C_{n,k}$ is 2-vertex globally rigid.

Theorem 13. *The graph $C_{n,k}$ is strongly minimally 2-vertex globally rigid for all $k \geq 2$ and $n \geq 2k+1$.*

It remains an open problem to find an inductive construction for the family of strongly minimally 2-vertex globally rigid graphs. Although 2-extensions alone are probably not sufficient to build up every graph in this family from a small set of base graphs, the following might be a useful partial result.

Conjecture 14. *Let $G = (V, E)$ be a 4-connected and essentially 6-edge connected 4-regular graph with $|V| \geq 9$. Then there exists a vertex $v \in V$ such that some 2-reduction at v yields a graph which is also 4-connected and essentially 6-edge connected.*

The complete bipartite graph $K_{4,4}$ shows that the lower bound on the size of the graph is essential.

6 2-vertex rigidity: a new construction

B. Servatius [15] showed that a strongly minimally 2-vertex rigid graph G on $|V|$ vertices has $2|V|-1$ edges for all $|V| \geq 5$, and that in such a graph exactly two vertices x, y are of degree three and the remaining vertices each have degree four. If x, y are non-adjacent, we say that G is of Type 1. Otherwise it is of Type 2. See Figure 5. Further results of [15] imply an inductive construction of this family of graphs in which the base graph is $K_5 - e$ and the operation used is 1-extension. In this section we establish a connection between essentially 6-edge connected graphs and 2-vertex rigid graphs, which makes it possible to obtain a new inductive construction for the latter family by using 2-extensions.

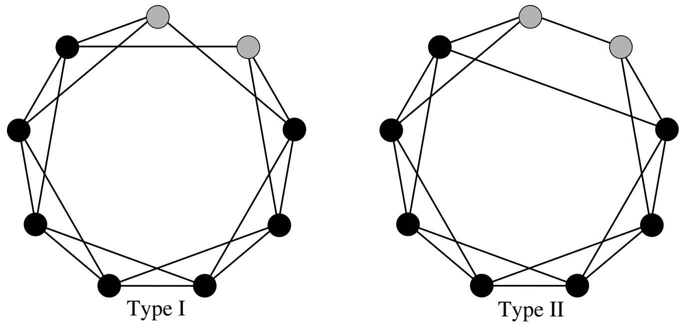

Figure 5: Strongly minimally 2-vertex rigid graphs on nine vertices.

We shall need the following corollary of [15, Theorem 3.1], which shows that a strongly minimally 2-vertex rigid graph is "nearly" 3-edge rigid.

Theorem 15. *[15] Let $G = (V, E)$ be a 2-vertex rigid graph with $|E| = 2|V| - 1$, and let $e, f \in E$. Then $G - \{e, f\}$ is rigid, unless (i) G is of Type 1, and e and f belong to the star of the same degree three vertex, or (ii) G is of Type 2, and e and f belong to the union of the stars of the degree three vertices.*

Lemma 16. *Let $G = (V, E)$ be a strongly minimally 2-vertex rigid graph with degree three vertices x, y. Then*
(i) if G is of Type 1 then $G' = G + xy$ is 4-regular and essentially 6-edge connected,
(ii) if G is of Type 2 then $G' = G/\{x, y\}$ is 4-regular and essentially 6-edge connected.

Proof. We prove (i) and (ii) simultaneously, since their proofs are similar. First observe that G' is 4-regular in both cases: it is obvious when G is of Type 1, and follows from the fact that x and y cannot have a common neighbour z when G is of Type 2, for otherwise $G - z$ is not rigid.

It remains to show that G' is essentially 6-edge connected. It is easy to see that G' is rigid, which implies, by 4-regularity and Lemma 9, that G' is essentially 4-edge connected. For a contradiction suppose that G' has a non-trivial edge cut $F' = \delta_{G'}(X')$ with $|F'| = 4$. The 4-regularity of G' and $|F'| = 4$ imply that we have $|X'| \geq 4$ and $|V(G') - X'| \geq 4$. This 4-edge cut defines an edge cut $F = \delta_G(X)$ in G with $|F| \leq 4$. (by splitting v, if $v \in X'$, or deleting xy, if $xy \in F$). We now observe that $|F| = 4$ must hold and the edges in F are pairwise disjoint: if this is not the case, then there is a vertex q for which $\delta(X - q)$ or $\delta(V - X - q)$ is an edge cut of size at most two in $G - q$ (namely, q is a common end-vertex of two incident edges in F, or an end-vertex of some edge in F, when $|F| = 3$). Since this edge cut is non-trivial and $G - q$ is rigid (by the 2-rigidity of G), it contradicts Lemma 9.

To finish the proof we pick two edges $e, f \in F$ which are not incident with x or y. We have four disjoint edges in F, so these edges indeed exist. By Theorem 15 $G'' = G - \{e, f\}$ is rigid. But $\delta_{G''}(X)$ is an edge cut of size two, contradicting Lemma 9. $\square$

Consider a vertex v in a 4-regular graph $G = (V, E)$ with $N_G(v) = \{a, b, c, d\}$ and replace it with two adjacent vertices v_1, v_2 of degree three such that v_1 is connected to a, b, and v_2 is connected to c, d. See Figure 6. We say that the resulting graph G' is obtained from G by *splitting* v.

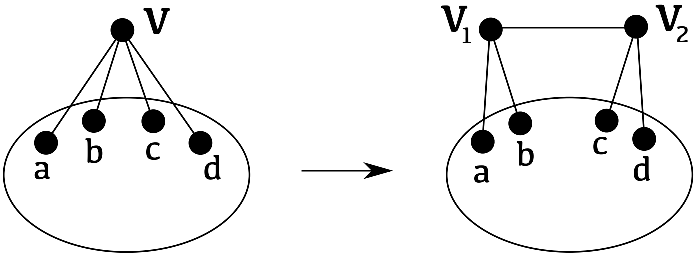

Figure 6: The splitting operation.

Lemma 17. *Let $G = (V, E)$ be an essentially 6-edge connected 4-regular graph and let G' be obtained from G by (i) deleting an edge $xy \in E$, or (ii) splitting a vertex $v \in V$ into two degree three vertices x and y. Then G' is 2-vertex rigid.*

Proof. (i) follows immediately from Lemma 11.

(ii) First we show that $G' - x$ (and hence, by symmetry, also $G' - y$) is rigid. Observe that $G' - x$ can be obtained from $G - v$ by a 0-extension. Since $G - v$ is rigid by Lemma 11, so is $G' - x$. Next consider a vertex $z \notin \{x, y\}$. Then, again by Lemma 11, $G - z$ is 2-edge rigid, and hence $G - z - e$ is rigid for any edge e incident with v. Since $G' - z$ can be obtained from $G - z - e$ by a 1-extension, it is also rigid. $\square$

Let K_6^- denote the graph obtained from K_5 by splitting a vertex. It is easy to check that the smallest strongly minimally 2-vertex rigid graph of Type 1 (resp. Type 2) is the graph $K_5 - e$ (resp. K_6^-). Since the 2-extension operation cannot turn a Type 1 graph into a Type 2 graph, we need different base graphs for the two types.

Theorem 18. *A graph $G = (V, E)$ with $|V| \geq 5$ is strongly minimally 2-vertex rigid if and only if either*
(i) G is of Type 1 and it can be obtained from $K_5 - e$ by a sequence of 2-extensions, or
(ii) G is of Type 2 and it can be obtained from K_6^- by a sequence of 2-extensions that do not involve the edge xy connecting the two degree three vertices of G and involve at most one edge incident with x or y.

Proof. The graphs $K_5 - e$ and K_6^- are strongly minimally 2-vertex rigid. To show sufficiency we prove that the 2-extension operations preserves (strongly minimal)

2-vertex rigidity. Let x, y denote the vertices of degree three in G and let e, f denote the edges involved in the 2-extension operation. Let the resulting graph be denoted by $G_{e,f}$.

First suppose that G is of Type 1. By Lemma 16(i) $G' = G + xy$ is essentially 6-edge connected and 4-regular. Thus performing a 2-extension on edges e, f in $G' = G + xy$ results in an essentially 6-edge connected and 4-regular graph $G'_{e,f}$. Since $G_{e,f} = G'_{e,f} - xy$, the claim follows from Lemma 17(i). Next suppose that G is of Type 2. Then we first we use Lemma 16(ii) to obtain a 4-regular essentially 6-edge connected graph G' by contracting the pair $\{x, y\}$. By our assumption, the edges corresponding to e and f are present in G' and share no end-vertex. Hence we can perform a 2-extension on e, f in G' to obtain a graph $G'_{e,f}$. Since $G_{e,f}$ can be obtained from $G'_{e,f}$ by a splitting operation, the claim follows from Lemma 17(ii).

We show necessity by induction on $|V|$. Let x, y be the two vertices of degree three in G. We shall assume that G is of Type 1. (The proof is similar when G is of Type 2.) For $|V| = 5$ the statement is obvious, as the only strongly minimally 2-vertex rigid graph on five vertices is $K_5 - e$. Suppose that $|V| \geq 6$. Let $G' = G + xy$. By Lemma 16(i) G' is essentially 6-edge connected and 4-regular. Lemmas 6 and 7 imply that either all vertices of G' are 2-reducible (when $|V| = 6$) or at least three vertices of G' are 2-reducible (when $|V| \geq 7$). Thus there is vertex $v \in V(G')$ which is different from x, y for which a suitable 2-reduction gives an essentially 6-edge connected 4-regular graph G'_v. Lemma 17(i) now implies that a 2-reduction at v in G gives rise to smaller 2-vertex rigid graph G_v. By induction, G_v can be obtained from $K_5 - e$ by a sequence of 2-extensions. This completes the proof, since G arises from G_v by a 2-extension operation. □

7 3-vertex rigidity

Motevallian, Yu, and Anderson [12] investigated the family of 3-vertex rigid graphs. They proved that such a graph $G = (V, E)$ with $|V| \geq 6$ satisfies $|E| \geq 2|V| + 2$, and suggested an inductive construction which can be used to obtain, from the base graph $K_6 - e$, a 3-vertex rigid graph on $|V|$ vertices, for each $|V| \geq 6$, for which $|E| = 2|V| + 2$ holds. Their construction is based on 2-extensions, executed on edge pairs satisfying certain conditions.

It was shown in [12, Section 4.3] that every 3-vertex rigid graph $G = (V, E)$ with $|E| = 2|V| + 2$ and $|V| \geq 6$ has exactly four vertices of degree five, and these vertices are pairwise adjacent. We say that $W = \{v \in V : d_G(v) = 5\}$ is the *core* of G. All the other vertices are of degree four. The smallest such graph is $K_6 - e$. They also showed that G contains two disjoint edges $ab, cd \in E$ with $a, c \notin W$, $b, d \in W$, and $ac \notin E$. We call such an edge pair *admissible*. The key statement [12, Theorem 4.14] is as follows.

Theorem 19. *Let $G = (V, E)$ be a 3-vertex rigid graph with $|V| \geq 6$ and $|E| = 2|V| + 2$, and let ab, cd be an admissible edge pair in G. Then the graph G' obtained from G by a 2-extension on edges ab, cd is 3-vertex rigid.*

Starting with $K_6 - e$ and repeatedly applying 2-extension operations on admissible edge pairs, one obtains an infinite family of 3-vertex rigid graphs with $2|V| + 2$ edges. Together with the matching lower bound, it shows that this is the tight edge count for the size of the strongly minimally 3-vertex rigid graphs.

However, the proof in [12] is incomplete. We present a complete proof in the Appendix. It turns out that in order to deal with an important case not covered by the proof in [12], we need an extension of Theorem 15 on removable edge pairs in strongly minimally 2-vertex rigid graphs (which have $2|V| - 1$ edges on $|V|$ vertices) to 2-vertex rigid graphs with $2|V|$ edges. This extension, which may be of independent interest, is proved in the next subsection.

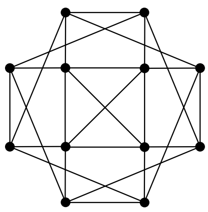

Figure 7: A strongly minimally 3-vertex rigid graph in which no two core vertices have a common neighbour which does not belong to the core.

We remark that not all strongly minimally 3-vertex rigid graphs can be obtained from $K_6 - e$ by a sequence of 2-extensions on admissible edge pairs. See Figure 7. We can also provide a direct construction of an infinite family of strongly minimally 3-vertex rigid graphs. Consider the graphs D_k, $k \geq 3$, with vertex set $\{a_i, b_i : 0 \leq i \leq k-1\}$ and edge set $\{a_i b_{i+1}, b_i a_{i+1} : 1 \leq i \leq k-1\} \cup \{a_0 b_0, a_1 b_1\}$, counting indices modulo k. See Figure 8. Note that $D_3 = K_6 - e$. It is easy to check these graphs are indeed 3-vertex rigid with $|E| = 2|V| + 2$.

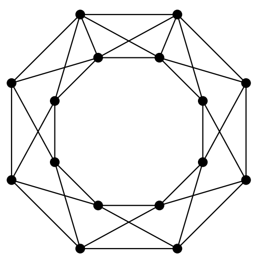

Figure 8: The graph D_8.

7.1 2-vertex rigidity revisited

Deleting two edges e, f from a 2-vertex rigid graph $G = (V, E)$ may destroy rigidity. It happens, for example, when the pair e, f belongs to a nontrivial 4-edge cut (cf. Lemma 9) or when e and f are incident with the same cubic subtree (cf. Lemma 3(ii)).

The next theorem asserts that there is no other case if $|E| = 2|V|$ holds. In this sense it is a direct extension of Theorem 15, which is valid in the case $|E| = 2|V| - 1$. The case $|E| = 2|V| - 1$ is much simpler: as we discussed earlier, exactly two vertices of G belong to cubic subtrees, and there is at most one non-trivial 4-edge cut in the graph. Moreover, if such a cut exists (this happens when G is of Type 2), then the edges of the non-trivial 4-edge cut are incident with the same cubic subtree (c.f. Lemma 16).

Theorem 20. *Let $G = (V, E)$ be a 2-vertex rigid graph with $|E| = 2|V|$ and let $e, f \in E$. Then one of the following holds:*
(i) $G - \{e, f\}$ is rigid,
(ii) there is a nontrivial 4-edge cut $\delta(X)$ in G with $\{e, f\} \subset \delta(X)$,
(iii) there is a cubic subtree $G[X]$ of G with $\{e, f\} \subset S_G(X)$.

Proof. Suppose that $G - \{e, f\}$ is not rigid. Let $H = G - e$. Since G is 2-vertex rigid, it is also 2-edge rigid. Hence, H is a rigid spanning subgraph of G with $2|V| - 1$ edges. Thus H can be obtained from a minimally rigid spanning subgraph B of G by adding two edges, say h_1, h_2. Let C_i be the unique rigidity circuit in $B + h_i$, for $i = 1, 2$. Now $H - f$ is not rigid, which implies, by using that rigidity circuits are 2-edge rigid, that the endvertices of f cannot belong to the same rigidity circuit C_i, $i = 1, 2$. Observe that we cannot have $|(V(C_1) \cap V(C_2)| = 1$. This follows from the fact that each rigidity circuit has minimum degree three, and, hence, the unique common vertex v satisfies $d_G(v) \geq 6$. But then $G - v$ has too few edges to be rigid, contradicting 2-vertex rigidity. Moreover, the gluing lemma implies that if $|(V(C_1) \cap V(C_2)| \geq 2$ holds, then $C_1 \cup C_2$ is rigid. We shall consider two cases.

Case 1: $V(C_1) \cup V(C_2) = V$.

If $|(V(C_1) \cup V(C_2)| \geq 2$, then $C_1 \cup C_2$ is a rigid spanning subgraph of H. Since $f \notin E(C_1) \cup E(C_2)$, it follows that $H - f$ is rigid, a contradiction. Thus, we may assume that $V(C_1) \cap V(C_2) = \emptyset$. Then $\delta_H(V(C_1))$ is a nontrivial edge cut in H. Let $q = |\delta_H(C_1)|$. The rigidity of G implies $q \geq 3$. Furthermore, we have $2|V| - 1 = |E(H)| \geq |E(C_1)| + |E(C_2)| + q = 2|V(C_1)| - 2 + 2|V(C_2)| - 2 + q = 2|V| - 4 + q$, which gives $q \leq 3$. On the other hand, the 2-edge rigidity of G implies $|\delta_G(V(C_1))| \geq 4$. Therefore, $\delta_G(V(C_1))$ is a nontrivial 4-edge cut in G with $\{e, f\} \subset \delta_G(V(C_1))$. Hence (ii) holds.

Case 2: $V(C_1) \cup V(C_2) \neq V$.

Let $X = V(G) \setminus (V(C_1) \cup V(C_2))$. Consider a vertex $x \in X$. The 2-vertex rigidity of G implies $d_G(x) \geq 3$. To see that we must have equality, suppose that $\deg_G(x) \geq 4$. Then $|E(G - x)| \leq 2|V(G - x)| - 2$, from which we obtain, by using the 2-vertex rigidity of G again, that $G - x$ is a rigid graph which contains at most one rigidity circuit. It contradicts the choice of x. Thus each vertex in X has degree three in G.

If $V(C_1) \cap V(C_2) = \emptyset$, then for every vertex $v \in V$ there exists a rigidity circuit in $G - v$. An argument similar to that of the previous paragraph gives that, in this case, we have $d_G(v) \leq 4$, for all $v \in V(G)$. Since $|E| = 2|V|$, it follows that G is 4-regular, which contradicts the existence of the degree-three vertices in X. Thus, we may assume that $|V(C_1) \cap V(C_2)| \geq 2$. So $C_1 \cup C_2$ is rigid.

Next we show that $G[X]$ is a cubic subtree of G. We have already verified that each vertex in X has degree three in G. Suppose that for some $Y \subseteq X$ the subgraph $G[Y]$ is a cycle. Then $s_G(Y) \leq 3|Y| - |Y| = 2|Y|$, which gives $s_{G-h}(Y) \leq 2|Y| - 1$ for any edge h of the cycle. Since $G - h$ is rigid, it contradicts Lemma 2. Thus, $G[X]$ is a forest. Let us assume that $G[X]$ is disconnected and consider a nontrivial partition $X = X_1 \cup X_2$ for which there is no edge in G from $G[X_1]$ to $G[X_2]$. Then the 2-edge rigidity of G and Lemma 2 yields:

$$s_G(X) = s_G(X_1) + s_G(X_2) \geq 2|X_1| + 1 + 2|X_2| + 1 = 2|X| + 2.$$

Hence, $|E(C_1 \cup C_2)| = |E| - s_G(X) \leq 2|V(C_1) \cup V(C_2)| - 2$ follows. But this contradicts that $C_1 \cup C_2$ is rigid and contains at least two rigidity circuits. Thus, $G[X]$ is a cubic subtree, as claimed.

Since $C_1 \cup C_2$ is rigid, $G - e$ is rigid, and $G - \{e, f\}$ is not rigid, we can now use Lemma 3 to deduce that $\{e, f\} \subset S_G(X)$. Hence (iii) holds. $\square$

8 Concluding remarks

In this paper we obtained a new inductive construction for the family of 4-regular essentially 6-edge connected graphs and showed that this family coincides with the families of 4-regular 3-edge rigid (resp. 2-edge globally rigid) graphs. We also characterized the 4-regular 2-vertex globally rigid graphs and proved several results on strongly minimally 2-vertex rigid and 3-vertex rigid graphs.

Families of 4-regular graphs satisfying certain connectivity or rigidity properties play an important role in other problems, too. For example, they show up in the characterization of rigid or globally rigid vertex transitive graphs [6], circuit decompositions of Eulerian graphs [2], and in massless ϕ^4-theory in physics [13]. Hence, our results (in particular, Theorem 8) may find further applications in the future.

9 Acknowledgements

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10 Appendix

We prove Theorem 19 in two parts. The first part, which deals with all but one of the subcases, is essentially the proof of [12, Theorem 4.14]. The second part completes the proof by using Theorem 20.

The first part of the proof of Theorem 19: Let z be the new vertex created by the 2-extension. We need to show that $G' - \{x, y\}$ is rigid for all $x, y \in V(G')$. We start with the cases when z is not one of the deleted vertices. Then $x, y \in V(G)$ and

3-vertex rigidity of G implies that $G - \{x, y\}$ is rigid. By symmetry, we have four cases to consider.

(a) $x, y \notin \{a, b, c, d\}$. Then $G' - \{x, y\}$ is obtained from $G - \{x, y\}$ by a 2-extension, and hence it is rigid.

(b) $x \in \{a, b\}$ and $y \notin \{a, b, c, d\}$. Then $G' - \{x, y\}$ is obtained from $G - \{x, y\}$ by a 1-extension, and hence it is rigid.

(c) $x \in \{a, b\}$ and $y \in \{c, d\}$. Then $G' - \{x, y\}$ is obtained from $G - \{x, y\}$ by a 0-extension, and hence it is rigid.

(d) $x = a, y = b$. Since $d_G(b) = 5$, the graph $G - b$ is a 2-vertex rigid graph with $|E(G - b)| = 2|V(G - b)| - 1$. Thus, $G - b$ has exactly two vertices of degree three, a and some other vertex a'. If $aa' \in E(G)$ then $ac \notin E$ implies $a' \neq c$. Let aq be an edge incident with a in $G - b$. We can now deduce from Theorem 15 that $G - b - \{aq, cd\}$ is rigid. Since the degree of a in this graph is equal to two, it follows that $G - \{b, a\} - cd$ is also rigid. Since G' is obtained from $G - \{b, a\} - cd$ by a 0-extension, it is rigid.

(e) Let us assume now that $x = z$. If $y \in \{a, b\}$ then $G' - \{z, y\} = G - y - cd$. Since $G - y$ is 2-vertex rigid, it is also 2-edge rigid. So $G' - \{z, y\}$ is rigid. The case $y \in \{c, d\}$ is similar. So we may suppose that $y \notin \{a, b, c, d\}$. Then $G' - \{z, y\} = G - y - ab - cd$.

If $d_G(y) = 5$, then $G - y$ is 2-vertex rigid with $|E(G - y)| = 2|V(G - y)| - 1$. In this case we can use Theorem 15 again to show that deleting the pair ab,cd preserves rigidity. If $yc \notin E$, then $d_{G-y}(c) = d_{G-y}(d) = 4$, and hence $G - y - ab - cd$ is rigid. The case $ya \notin E$ is similar. If $yc, ya \in E$ then, since $ac \notin E$, $G - y$ has two non-adjacent vertices of degree three: a and c. Hence $G - y - ab - cd$ is rigid. This completes the subcase when $d_G(y) = 5$.

The second part of the proof of Theorem 19: It remains to consider the case when $d_G(y) = 4$. Let $H = G - y$. Since G is 3-vertex rigid and $d_G(y) = 4$, H is 2-vertex rigid with $|E(H)| = 2|V(H)|$. Let $e = ab$, $f = cd$. We need to show that $H - \{e, f\}$ is rigid. By Theorem 20, it suffices to show that no nontrivial 4-edge cut or cubic subtree of H contains both edges.

First, suppose that there is a nontrivial 4-edge cut $\delta_H(X)$ with $\{e, f\} \subset \delta_H(X)$. The 2-vertex rigidity of H implies that the edges in $\delta_H(X)$ are pairwise disjoint. Thus, W lies entirely in one of two sides of the cut, say $W \subseteq V \setminus X$. Hence $a, c \in X$. Since G is 3-edge rigid and each vertex of X has even degree in G, it follows that y has exactly two neighbors in X as well as in $V \setminus X$. Thus $d_G(X) = d_G(V \setminus X) = 6$. This shows that $|X| \geq 3$ holds, for otherwise $ac \notin E$ implies $d_G(X) = d_G(a) + d_G(c) \geq 8$, a contradiction. Let $q \in X$ with $yq \notin E$. Then $d_H(q) = 4$ holds and, hence, $H - q$ satisfies $|E(H - q)| = 2|V(H - q)| - 2$. Now $H - q$ is rigid, so it can be obtained from a sparse graph by adding an edge. On the other hand $W \subseteq V \setminus X$ implies $i_{H-q}(V - X) \geq \frac{4|V-X|+4-6}{2} = 2|V - X| - 1$, a contradiction.

Next, suppose that there is a cubic subtree $H[X]$ in H with $\{e, f\} \subset S_G(X)$. Then we must have $d_H(a) = d_H(c) = 3$ and $\{a, c\} \subseteq X \subseteq N_G(y)$. Moreover, since $ac \notin E$, we have $|X| \geq 3$. This implies that there exist two incident edges $jk, kl \in G[X]$. Therefore k and y are two adjacent degree-four vertices in G with two common neighbours: j and l. But then $s_{G-j-l}(\{y, k\}) = 3$, from which Lemma 2 gives that $G - y - k$ is not rigid, a contradiction. QED.

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